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The Theory of Surface Wave Diffraction by Symmetric Crustal Discontinuities

by

JULIUS KANE

and

JOHN SPENCE

University of Rhode Island
Kingston, Rhode Island

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Summary

A major barrier in comparing seismic theory with observed wave trains, stems from the fact that elastic wave characteristics are influenced by discontinuities along the propagation path, and any understanding of such signal corruption would require a knowledge of the diffracted fields at the appropriate obstacles. However, even a relatively simple crustal feature such as a discontinuous change in terrain presents major difficulties if the relevant problem of diffraction in a wedge-shaped region is considered. Although some first-order calculations have been made by Lapwood (1961), Kane and Spence (1963), and Hudson and Knopoff (1963), the theoretical discussion of the diffraction effects are hampered by the intractability of the associated boundary value problems. In this report, we show how one can take advantage of symmetry considerations and variational techniques to rapidly estimate reflection, transmission, and conversion coefficients for elastic wave diffraction at symmetric wedge-shaped obstacles. In Part I, we illustrate the ideas by a discussion of the vector problem of Rayleigh wave propagation along the faces of an elastic wedge with free boundaries. In Part II, we analyze the scalar problem of multi-mode Love wave diffraction in a symmetric layered wedge.

PART I. RAYLEIGH WAVES ON AN ELASTIC WEDGE

1. Fundamental equations

The tremors $\vec{s}(u,v,w)$ of an elastic solid characterized by the Lamé parameters λ, μ, and density ρ, can be derived from a scalar potential $\Phi(x,y,z)$, and a vector potential $\Psi[\psi_x(x,y,z),\ \psi_y(x,y,z),\ \psi_z(x,y,z)]$ by the relation

$$\vec{s}(u,v,w) = \nabla\Phi + \nabla \times \Psi . \qquad (1.1)$$

For two-dimensional motions which are independent of the z-coordinate, both Φ and Ψ are but functions of x and y, or $\Phi = \Phi(x,y)$, and $\Psi = \Psi(x,y)$. Furthermore, we can neglect pure distortions by setting $\psi_x = \psi_y = 0$, so that the vector potential $\Psi = \Psi[0,\ 0,\ \psi_z(x,y)]$ is characterized by one scalar component and the subscript on ψ_z can be dropped without confusion. If we assume that the vibrations are harmonic, we can suppress a time factor $e^{-i\omega t}$, and it can be shown that Φ, and ψ the z-component of the vector potential, satisfy the reduced wave equations

$$(\nabla^2 + k_c^2)\,\Phi = 0, \qquad k_c^2 = \omega^2\rho/(\lambda+2\mu), \qquad (1.2)$$

$$(\nabla^2 + k_s^2)\,\psi = 0, \qquad k_s^2 = \omega^2\rho/\mu . \qquad (1.3)$$

Once the potentials Φ and ψ are known, the displacement vector $\vec{s}$ is given by (1.1), and the resultant stress dyadic $\overline{\overline{\sigma}}(\Phi,\psi)$ can be given in symbolic notation as

$$\mathfrak{S}(\phi,\psi) = \lambda \mathfrak{J}\, \nabla\cdot\dot{\vec{s}} + \mu\,(\nabla\vec{s} + \vec{s}\nabla) \tag{1.4}$$

where $\mathfrak{J}$ is the unity dyadic, the idemfactor.

2. The Rayleigh wave potentials

A time harmonic Rayleigh wave, or R-wave for brevity, is comprised of a pair of exponential solutions of (1.2) and (1.3), $\omega = r(\phi_R, \psi_R)$. If the elastic solid lies within the half-space $y \le 0$ (cf. Figure 1a), then these solutions, in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, assume the form

$$R(\phi_R, \psi_R) = \begin{cases} \phi_R = e^{\gamma(\theta)\Xi} & (1.5) \\ \psi_R = -i\,[illegible]\, e^{\beta(\theta)\Xi} & (1.6) \end{cases}$$

wherein the exponential variation is given by

$$\gamma(\theta) = \sqrt{1-(v_R/v_c)^2}\,\sin\theta + i\cos\theta, \quad [illegible] \tag{1.7}$$

$$\beta(\theta) = \sqrt{1-(v_R/v_s)^2}\,\sin\theta + i\cos\theta, \quad v_s^2 = [illegible] \tag{1.8}$$

and [illegible], the magnitude of the ratio of the shear potential coefficient of ψ_R to the compressional one ϕ_R is

$$= \frac{2\sqrt{1-(v_R/v_c)^2}}{2-(v_R/v_s)^2}\,. \tag{1.9}$$

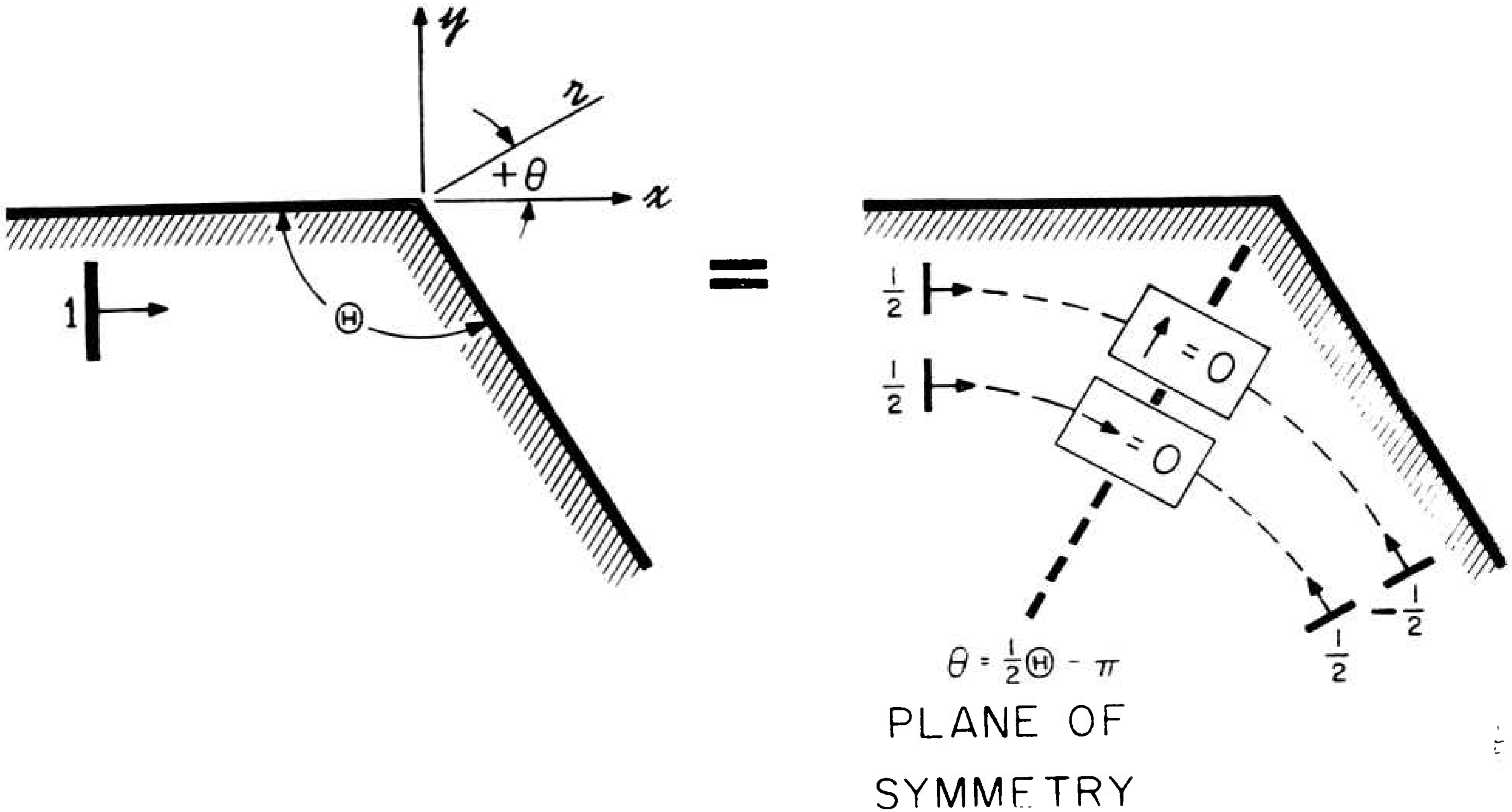

Figure 1: A unit Rayleigh Wave incident along one face of an elastic wedge is equivalent to four partial waves. The partial waves of like parity comprise the excitation for the even problem, and the partial waves of unlike parity furnish the initial disturbance for the odd problem.

and ξ is a dimensionless distance parameter

$$\xi = k_R r = \frac{\omega}{v_R} r. \tag{1.10}$$

The parameters v_c and v_s represent the velocities of the compressional and shear body waves respectively. For a given Poisson's ratio ν

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \tag{1.11}$$

one needs choose the Rayleigh wave velocity v_R, which is less than v_s, so that the stresses induced by ϕ_R and ψ_R vanish at the surface $y = 0$

$$\Sigma(\phi_R, \psi_R) = 0, \qquad y = 0. \tag{1.12}$$

With this choice of v_R, the ϕ_R and ψ_R given by (1.5) and (1.6) are the vector and scalar potentials characterizing a Rayleigh wave traveling to the right with unit amplitude, (we shall speak of the coefficient of the compressional potential as the amplitude). It is very convenient to note that <u>we can reverse the direction of any harmonic wave by the operation of complex conjugation</u>; thus

$$R\,\Phi(\phi^*_R, \psi^*_R) = \begin{cases} R\,\phi^*_R = R\,e^{\rho^*(\theta)} & (1.13) \\[2ex] R\,\psi^*_R = -iR^-\,e^{\sigma^*(\theta)} & (1.14) \end{cases}$$

represents a Rayleigh wave traveling in the opposite direction with amplitude R.

3. Formulation of the boundary value problems

a. The major problem

The conundrum posed by Rayleigh wave diffraction in a wedge-shaped region is to find additional solutions ϕ_d and ψ_d, of (1.2) and (1.3) which represent diffracted fields in the interior of the wedge such that the stress dyadic $\Sigma(\phi_R + \phi_d, \psi_r + \psi_d)$ vanishes on both faces of the wedge for R-wave excitation along one face. In our geometry, (cf. Figure 1a), the R-wave is incident from infinity along the negative x-axis. We shall be principally concerned with calculating the complex amplitudes of the reflected and transmitted R-waves as a function of the wedge angle Θ and Poisson's ratio σ. This task can be substantially eased by reducing the major problem to a pair of minor problems involving even and odd symmetries. Consider the incident R-wave along the left wedge face to be the sum of two waves, each of half amplitude. Likewise, the absence of any excitation along the right wedge face is equivalent to a pair of incident R-waves along it, each of half amplitude, but opposed in sign. (cf. Figure 1b). The four partial R-waves can be separated into two groups: First, a pair of R-waves on either face of like parity which serves as the excitation of what we call the even minor problem. Second, another pair of R-waves whose amplitudes are of unlike parity which constitutes the excitation of the odd minor problem. If we designate the diffracted fields of the even and odd problems by the subscripts e and o respectively, then since we are dealing with linear equations, the desired major potentials ϕ_d and

ψ_d can be expressed as a superposition of the minor potentials

$$\phi_d = \phi_e + \phi_o , \tag{1.15}$$

$$\psi_d = \psi_e + \psi_o , \tag{1.16}$$

and likewise the displacement vector $\vec{s} = \vec{s}_e + \vec{s}_o$ can be decomposed into even and odd components.

b. The even minor problem

In the even problem, the wedge will suffer only even displacements* $\vec{s}_e$ about the plane of symmetry, and as a result, there can be no component of normal displacement along the plane of symmetry at $\theta = \Theta/2 - \pi$. It follows that the even problem is equivalent to finding the potentials in a bisected wedge with one face free of stresses which supports the incident Rayleigh wave, and the other face so constrained that the normal displacement vanishes there. That is, we seek solutions ϕ_e and ψ_e of (1.2) and (1.3) in a wedge of half-angle $\Theta/2$

EVEN

$$\left\{\begin{array}{ll} \Xi\,(\phi_R + \phi_e, \psi_r + \psi_e) = 0, & \theta = \pi \qquad (1.17) \\ \frac{1}{r}\frac{\partial}{\partial\theta}(\phi_R + \phi_e) + \frac{\partial}{\partial r}(\psi_R + \psi_e) = 0, & \theta = \Theta/2 - \pi \qquad (1.18) \end{array}\right.$$

c. The odd minor problem

By the same arguments, the odd problem which involves ϕ_o, ψ_o is a complementary version in the bisected wedge, wherein the tangential

*N.B. The displacements, and the compressional potential will be even about the plane of symmetry, but the shear potential will be an odd function, and vice versa for the odd problem.

displacements must vanish identically along the plane of symmetry, i.e.,

$$\text{ODD}\begin{cases}\mathfrak{S}(\phi_R+\phi_0,\ \psi_R+\psi_0)=0 & \theta=\pi \qquad (1.19)\\[2ex] \dfrac{\partial}{\partial r}(\psi_R+\psi_0)-\dfrac{1}{r}\dfrac{\partial}{\partial\theta}(\phi_R+\phi_0)=0 & \theta=\Theta/2-\pi \qquad (1.20)\end{cases}$$

d. The Reflection Coefficients

For either the even or the odd problem, the solution will contain a reflected Rayleigh wave. Let $\rho_e(\Theta/2,\sigma)$ and $\rho_o(\Theta/2,\sigma)$ be the complex ratios of the reflected to the incident Rayleigh wave amplitude for the even and odd minor problems in the bisected wedge. The reflection coefficient $R(\Theta,\sigma)$ for the original major problem will be

$$R(\Theta,\sigma)=\tfrac{1}{2}\left[\rho_e(\Theta/2,\sigma)+\rho_o(\Theta/2,\sigma)\right], \tag{1.21}$$

and likewise the overall transmission coefficient $T(\Theta,\sigma)$ will be

$$T(\Theta,\sigma)=\tfrac{1}{2}\left[\rho_e(\Theta/2,\sigma)-\rho_o(\Theta/2,\sigma)\right]. \tag{1.22}$$

Formulas (1.21) and (1.22) can be verified by a glance at Figure 1b which indicates that the overall reflection coefficient R results from a superposition of the partial reflection coefficients $\frac{1}{2}(\rho_e+\rho_o)$, and the transmission coefficient T from their interference $\frac{1}{2}(\rho_e-\rho_o)$.

4. The Variational Principle

a. Discussion

Variational procedures consist of assuming a suitable trial function containing unspecified coefficients, and then choosing these parameters to minimize certain quantities. One major advantage of the variational method is that first-order accuracy in the trial function usually gives results which are accurate to second-order, because of the stationary character of the approximation.

A natural aperture in the present problem is the plane of symmetry and we can assume it to be illuminated by an incident $\mathcal{R}$-wave, and a reflected one with an adjustable amplitude. In the even problem, the net angular displacement must vanish along the plane of symmetry. A unit $\mathcal{R}$-wave traveling to the right gives rise to the angular component of the displacement

$$s_\theta^{inc} = k_R \left[\frac{\partial \alpha}{\partial \theta} e^{\gamma z} + \beta e^{\beta z} \right], \tag{1.23}$$

and likewise an $\mathcal{R}$-wave of amplitude ρ_e moving to the left generates the disturbance

$$\rho_e s_\theta^{ref} = \rho_e (s_\theta^{inc})^* \tag{1.24}$$

which apart from an amplitude factor is the complex conjugate of (1.23). Only if there is no discontinuity, or if $\Theta = \pi$ can we make the angular displacement of the trial function

$$s_\theta^T = s_\theta^{inc} + \rho_e (s_\theta^{inc})^* \tag{1.25}$$

vanish for all r along the plane of symmetry by properly choosing ρ_e. Otherwise $\alpha \neq \alpha^*$, $\beta \neq \beta^*$, and no choice of ρ_e can make s_θ^T

vanish at more than an isolated set of points. There are at least two ways by which we can improve matters: We could use a more complex trial function which acknowledges body-wave contributions to the diffracted field, or, since the residual displacement s_θ^T is explicitly known, we can use it as the aperture illumination of a Green's theorem type calculation to correct the variational estimate. However, the practical seismic interest is in the realm of small discontinuities, and for this case we shall see that the elementary trial function yields satisfactory results.

b. Definition of the scalar product

While there are many guages by which s_θ^T can be minimized, we shall choose a ρ_e which minimizes s_θ^T in the mean square sense. For this purpose, let us define the complex scalar product of two functions $u(r,\theta)$ and $v(r,\theta)$ to be

$$(u,v) \equiv \int_0^\infty u(r,\theta)\, v^*(r,\theta)\, dr, \tag{1.26}$$

where the integration is to be carried out along the plane of symmetry $\theta = \Theta/2 - \pi$. To each complex function $u(r,\theta)$ we can attach a positive definite number, the norm of u or $\| u \|$ which is defined to be $\sqrt{(u,u)}$. The norm $\| u \|$ depends on the wedge angle, and is to be distinguished from $u^2 = (u,u^*)$ which is in general complex.

c. The Even Subsidiary Reflection Coefficient

With this notation, the mean square value of the angular displacement of the trial function s_θ^T is

$$\| s_\theta^T \|^2 = \left(s_\theta^{inc} + \rho_e (s_\theta^{inc})^*,\ s_\theta^{inc} + \rho_e (s_\theta^{inc})^* \right), \tag{1.27}$$

and this will be a [illegible] if and only if ρ_e is chosen as

$$\rho_e(\Theta/2,\cdot) = \frac{(E_\theta^{inc})^2}{\|E_\theta^{inc}\|^2}, \qquad (1.28)$$

or explicitly in terms of $\alpha(\theta)$, $\beta(\theta)$ and [illegible],

$$\rho_e(\Theta/2,\cdot) = \frac{\frac{1}{2\alpha}\left(\frac{\partial\alpha}{\partial\theta}\right)^2 + \frac{\beta^2}{2} - \frac{2\,[\text{illegible}]}{\alpha+\alpha^*}\left(\frac{\partial\alpha}{\partial\theta}\right)}{\frac{1}{\alpha+\alpha^*}\left|\frac{\partial\alpha}{\partial\theta}\right|^2 + \frac{|\beta|^2}{\beta+\beta^*} + i\Gamma\left[[\text{illegible}]\frac{\partial\alpha}{\partial\theta} - \frac{\beta^*}{\beta+\beta^*}\left(\frac{\partial\alpha}{\partial\theta}\right)\right]} \qquad (1.29)$$

d. <u>The Odd Subsidiary Reflection Coefficient</u>

If we use an analogous trial function, and similar reasoning, we find that if ρ_o is to be an optimal choice we need make the selection

$$\rho_o(\Theta/2,\cdot) = \frac{(s_r^{inc})^2}{\|s_r^{inc}\|^2}, \qquad (1.30)$$

or

$$\rho_o(\Theta/2,\cdot) = \frac{\frac{\beta^2}{2\alpha} + \frac{[\text{illegible}]^2}{2\alpha}\left(\frac{\partial\alpha}{\partial\theta}\right)^2 + [\text{illegible}]\left(\frac{\partial\alpha}{\partial\theta}\right)}{\frac{|\beta|^2}{\alpha+\alpha^*} + [\text{illegible}]\left|\frac{\partial\alpha}{\partial\theta}\right|^2 - i\Gamma\left[\frac{\beta^*}{[\text{illegible}]}\left(\frac{\partial\alpha}{\partial\theta}\right) [\text{illegible}]\left(\frac{\partial\alpha}{\partial\theta}\right)\right]} \qquad (1.31)$$

5. <u>Discussion [illegible] overall reflection and transmission coefficients</u>

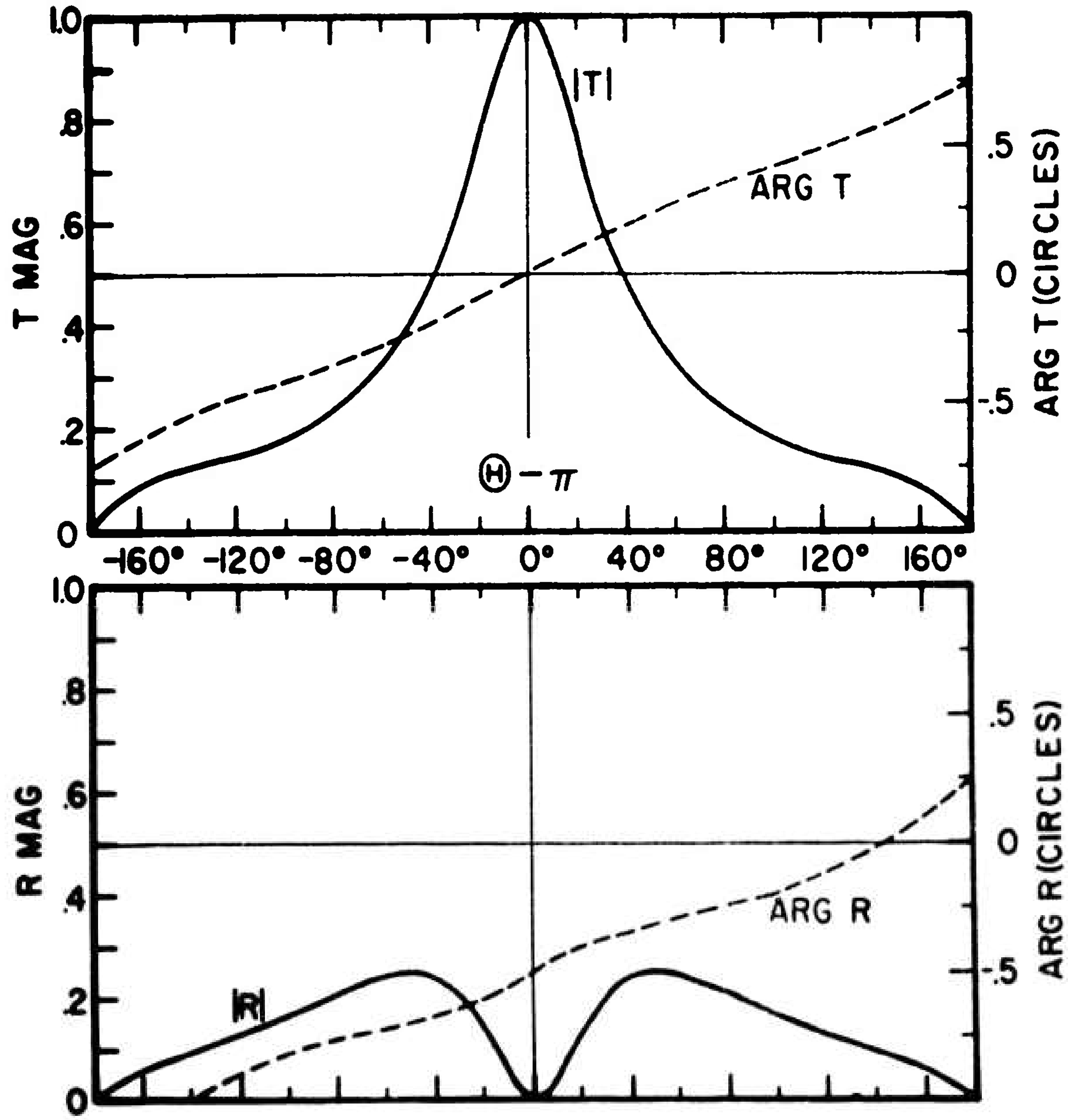

Figure 2: The diffraction coefficients R and T for a trial function consisting of but an incident and reflected wave. In this case, Poisson's ratio $\sigma = 1/4$.

With an explicit ζ_e and ζ_o at our disposal, we can evaluate the $R(\Theta,\nu)$ and $T(\Theta,\nu)$ germane to our elementary trial function. For reference, their complex variation is plotted as a function of the discontinuity angle $\Theta-\pi$, in Figure 2 for Poisson's ratio $\nu = 1/4$. The present magnitudes $|T|$ are somewhat smaller than that given by earlier first-order calculations (Kane and Spence 19[illegible]), which do not simultaneously yield R and T. Since we evaluate both R and T together, we must, in effect, withdraw some energy from the transmitted field to allow for the reflected wave. Furthermore, it is the nature of the variational technique to underestimate the subsidiary diffraction coefficients ζ_e and ζ_o since it only yields their projection in the sub-space spanned by the trial function.

Although the analysis is certainly valid for a small enough discontinuity in wedge angle, the utility of the procedure can not be established until there is some estimate of the errors committed. A feature of the present procedure is that it suggests a natural guage of the accuracy. While ζ_e and ζ_o are so chosen that $\| s_0^T \|$ and $\| s_r^T \|$ are minimized, both s_0^T and s_r^T are non-zero along the aperture plane. These residuals, which are explicitly known, can not be farther reduced without introducing new features such as body wave contributions into the analysis. Since $[1-|R|^2-|T|^2]$ represents that fraction of energy unaccounted for, we can estimate the need for improving the calculations by examining this quantity.

<u>This error estimate is a very generous one</u> because only part of it implies higher-order corrections to R and T, the remainder representing energy which is accounted for by R-wave to body wave conversion. The data of Figure 2 shows that if $|\Theta - \pi| < 10^\circ$, the present analysis accounts for at least 92 percent of the energy, and therefore the theory should not require further improvement within this range. We can also compare the present theory with experiment, but we must be very careful if we do so, because there are fundamental distinctions between analysis in the harmonic domain and pulse measurements (*cf.* Appendix).

PART II: LOVE WAVES ON AN ELASTIC WEDGE

1. Introduction

A layered solid can support surface waves which are horizontally polarized shear waves trapped in the superficial layer. Since these Love waves, as they are known, have no compressional component, it is not necessary to introduce potentials, and it is possible to work directly with one scalar function $w(x,y)$, the z-component of the displacement vector

$$\vec{s} = [0,\ 0,\ w(x,y)]. \tag{2.1}$$

Within the E_1 layer, w satisfies the wave equation

$$(\nabla^2 + k_1^2)\ w(x,y) = 0, \qquad k_1^2 = \omega^2\rho_1/\mu_1, \tag{2.2}$$

and within the E_2 substrate, w obeys

$$(\nabla^2 + k_2^2)\ w(x,y) = 0, \qquad k_2^2 = \omega^2\rho_2/\mu_2. \tag{2.3}$$

At the free surface of E_1, the stress dyadic $\mathfrak{S}(\vec{s})$ must vanish, which will be true provided that the normal derivative

$$\frac{\partial w}{\partial n} = 0 \tag{2.4}$$

vanishes there.

We assume the E_1-E_2 interface to be welded so that the displacement and normal stress must be continuous across this boundary. Along wedge face $\underline{A}$, these conditions will be satisfied provided that the Love

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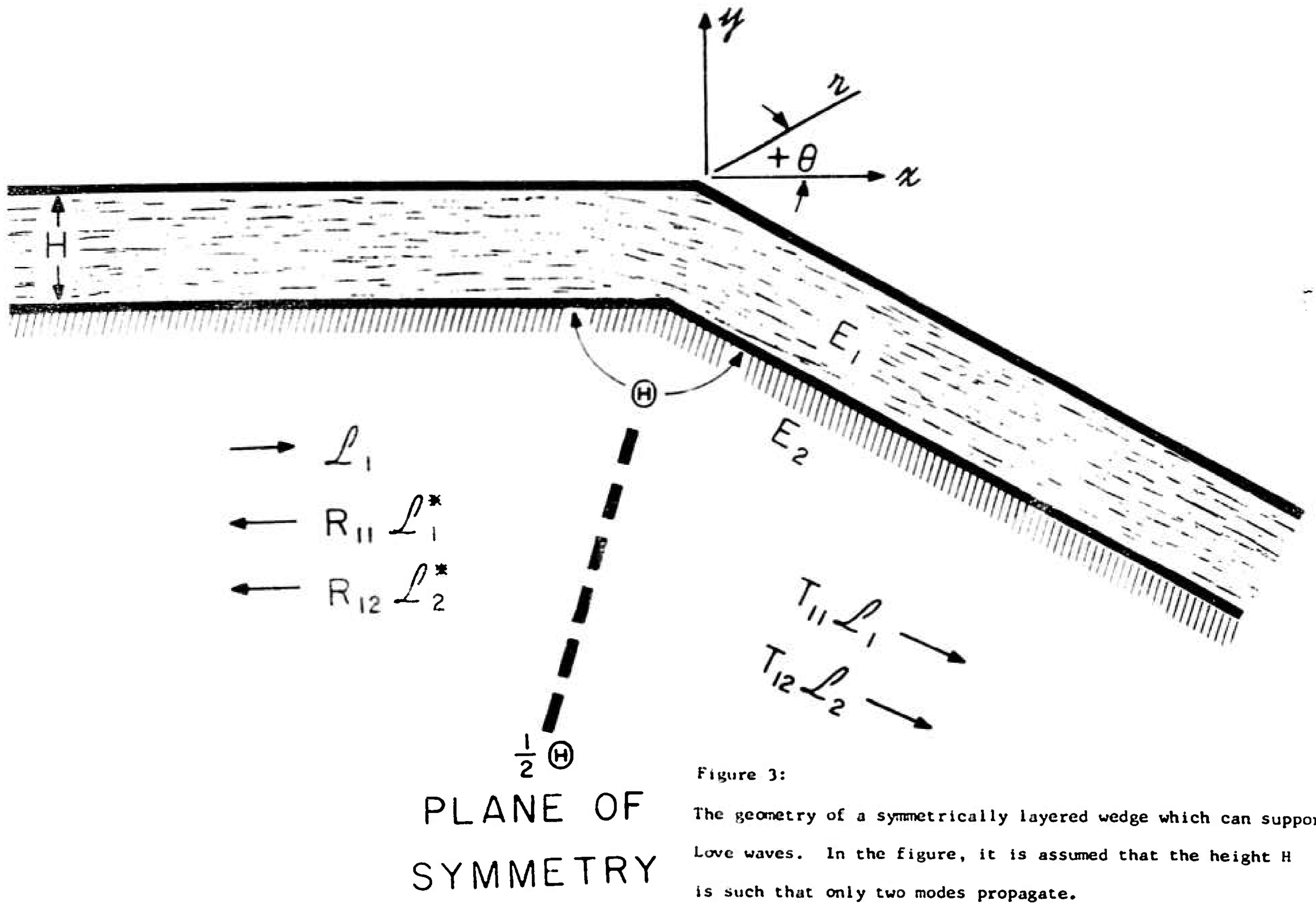


Figure 3:

The geometry of a symmetrically layered wedge which can support Love waves. In the figure, it is assumed that the height H is such that only two modes propagate.

waves, or Ω_i-waves of amplitude A_i have the form

$$A_i\Omega_i = \begin{cases} A_i \cos\sqrt{k_1^2 - \lambda_i^2}\, y \; e^{i\lambda_i x}, & 0 \geq y \geq -H \qquad (2.5) \\ A_i \cos\sqrt{k_1^2 - \lambda_i^2}\, H \exp\sqrt{\lambda_1^2 - k_2^2}\, y \; e^{i\lambda_i x}, & y \leq -H, \qquad (2.6) \end{cases}$$

and the propagation constants λ_i are the real roots of the period equation

$$\tan\sqrt{k_1^2 - \lambda_i^2}\, H = \frac{\mu_2}{\mu_1}\sqrt{\frac{\lambda_i^2 - k_2^2}{k_1^2 - \lambda_i^2}} \qquad (2.7)$$

If the shear wave is to be trapped in the layer, or if the Love wave is to propagate, we need $|k_2| \leq |\lambda_i| \leq |k_1|$. For any thickness however small, there is at least one root λ_1 corresponding to an acceptable solution --- the fundamental Ω_1-wave. As the acoustic thickness k_1H increases, other modes can propagate. In our discussion, we shall assume that the thickness is such that only two modes propagate, the fundamental, and one harmonic: the Ω_2-wave with a propagation constant λ_2. The analysis proceeds in a similar fashion if an arbitrary number of modes can propagate.

2. Formulation of the boundary value problem

We assume that an Ω_1- wave is incident along one face of a symmetrically layered wedge. At the discontinuity, four surface waves will be excited: A reflected and transmitted Ω_1- wave with amplitude coefficients R_{11} and T_{11} respectively, and reflected and transmitted Ω_2 waves whose amplitudes are the conversion coefficients R_{12} and T_{12} respectively. Our task will be to determine these diffraction coefficients as functions of the wedge angle Θ, the layer thickness H, and the elastic constants ρ_1, ρ_2, μ_1, μ_2.

By the same argument as in Part I, we can add and subtract a symmetric $\mathcal{S}_1$-excitation on the right wedge face which leads us to consider a pair of even and odd problems in a bisected wedge. Since Love wave diffraction is a scalar problem, the subsidiary boundary conditions along the aperture or plane of symmetry are simply

$$\text{EVEN:} \quad \frac{1}{r}\frac{\partial w}{\partial \theta} = 0, \qquad \theta = \Theta/2 - \pi \tag{2.8}$$

$$\text{ODD:} \quad w = 0, \qquad \theta = \Theta/2 - \pi \tag{2.9}$$

for the even and odd problems. The trial function will consist of an incident $\mathcal{S}_1$-wave, and reflected $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$-waves, with unknown amplitudes. If we denote the subsidiary reflection and conversion coefficient for the even problem in the bisected wedge as r_{11}^e and r_{12}^e, and similarly r_{11}^0 and r_{12}^0 for the odd problem, then the desired major coefficients are

$$R_{11} = \tfrac{1}{2}\,(r_{11}^e + r_{11}^0), \tag{2.10}$$

$$R_{12} = \tfrac{1}{2}\,(r_{12}^e + r_{12}^0), \tag{2.11}$$

$$T_{11} = \tfrac{1}{2}\,(r_{11}^e - r_{11}^0), \tag{2.12}$$

$$T_{12} = \tfrac{1}{2}\,(r_{12}^e - r_{12}^0). \tag{2.13}$$

As in Part I, we shall determine these coefficients by a variational procedure which ignores body wave contributions.

3. Solution

In the odd problem we shall choose r_{11}^{o} and r_{12}^{o} so that the residual variation $\varepsilon(r)$ along the aperture plane

$$\varepsilon(r) = \Omega_1 + r_{11}^{o}\,\Omega_1^{*} + r_{12}^{o}\,\Omega_1^{*}, \qquad \theta = \Theta/2 - \pi \tag{2.14}$$

is as small as possible in the mean square sense. Using the same definition of scalar product as in Part 1, we have

$$\| \varepsilon(r) \|^2 = \left(\Omega_1 + r_{11}^{o}\,\Omega_1^{*} + r_{12}^{o}\,\Omega_2^{*},\ \Omega_1 + r_{11}^{o}\,\Omega_1^{*} + r_{12}^{o}\,\Omega_2^{*}\right), \tag{2.15}$$

and this expression will be a minimum if and only if r_{11}^{o} and r_{12}^{o} satisfy the normal equations

$$(\Omega_1,\Omega_1) + r_{11}^{o}\,(\Omega_1^{*},\Omega_1) + r_{12}^{o}\,(\Omega_2^{*},\Omega_1) = 0, \tag{2.16}$$

$$(\Omega_1,\Omega_2) + r_{11}^{o}\,(\Omega_1^{*},\Omega_2) + r_{12}^{o}\,(\Omega_2^{*},\Omega_2) = 0. \tag{2.17}$$

Equations (2.16) and (2.17) can immediately be solved for r_{11}^{o} and r_{12}^{o}

$$r_{11}^{o} = -\frac{1}{\mathrm{DET}^{o}} \begin{vmatrix} (\Omega_1,\Omega_1) & (\Omega_2^{*},\Omega_1) \\ (\Omega_1,\Omega_2) & (\Omega_2^{*},\Omega_2) \end{vmatrix} . \tag{2.18}$$

and

$$r_{12}^{o} = -\frac{1}{\mathrm{DET}^{o}} \begin{vmatrix} (\Omega_1^{*},\Omega_1) & (\Omega_1,\Omega_1) \\ (\Omega_1^{*},\Omega_2) & (\Omega_1,\Omega_2) \end{vmatrix} . \tag{2.19}$$

where

$$\mathrm{DET}^o = \begin{vmatrix} (\Omega_1^*, \Omega_1) & (\Omega_2^*, \Omega_1) \\ (\Omega_1^*, \Omega_2) & (\Omega_2^*, \Omega_2) \end{vmatrix} . \tag{2.20}$$

In the same fashion, for the even problem we need choose r^e_{11} and r^o_{12} as

$$r^e_{11} = -\frac{1}{\mathrm{DET}^e} \begin{vmatrix} \left(\frac{1}{r}\frac{\partial \Omega_1}{\partial \theta}, \frac{1}{r}\frac{\partial \Omega_1}{\partial \theta}\right) & \left(\frac{1}{r}\frac{\partial \Omega_2^*}{\partial \theta}, \frac{1}{r}\frac{\partial \Omega_1}{\partial \theta}\right) \\ \left(\frac{1}{r}\frac{\partial \Omega_1}{\partial \theta}, \frac{1}{r}\frac{\partial \Omega_2}{\partial \theta}\right) & \left(\frac{1}{r}\frac{\partial \Omega_2^*}{\partial \theta}, \frac{1}{r}\frac{\partial \Omega_2}{\partial \theta}\right) \end{vmatrix} , \tag{2.21}$$

and

$$r^e_{12} = -\frac{1}{\mathrm{DET}^e} \begin{vmatrix} \left(\frac{1}{r}\frac{\partial \Omega_1^*}{\partial \theta}, \frac{1}{r}\frac{\partial \Omega_1}{\partial \theta}\right) & \left(\frac{1}{r}\frac{\partial \Omega_1}{\partial \theta}, \frac{1}{r}\frac{\partial \Omega_1}{\partial \theta}\right) \\ \left(\frac{1}{r}\frac{\partial \Omega_1^*}{\partial \theta}, \frac{1}{r}\frac{\partial \Omega_2}{\partial \theta}\right) & \left(\frac{1}{r}\frac{\partial \Omega_1}{\partial \theta}, \frac{1}{r}\frac{\partial \Omega_2}{\partial \theta}\right) \end{vmatrix} , \tag{2.22}$$

where

$$\mathrm{DET}^e = \begin{vmatrix} \left(\frac{1}{r}\frac{\partial \Omega_1^*}{\partial \theta}, \frac{1}{r}\frac{\partial \Omega_1}{\partial \theta}\right) & \left(\frac{1}{r}\frac{\partial \Omega_2^*}{\partial \theta}, \frac{1}{r}\frac{\partial \Omega_1}{\partial \theta}\right) \\ \left(\frac{1}{r}\frac{\partial \Omega_1^*}{\partial \theta}, \frac{1}{r}\frac{\partial \Omega_2}{\partial \theta}\right) & \left(\frac{1}{r}\frac{\partial \Omega_2^*}{\partial \theta}, \frac{1}{r}\frac{\partial \Omega_2}{\partial \theta}\right) \end{vmatrix} . \tag{2.23}$$

With this knowledge, the reflection, transmission and conversion coefficients are given by (2.10) - (2.13).

C. <u>Discussion of the results</u>

1. <u>Numerical Data</u>

We have used the preceding formulas to calculate the diffraction coefficients for an E_1-layer and E_2-substrate for which $k_1/k_2 = 1.297$ and $\mu_2/\mu_1 = 2.159$. The phase and group velocities for this case have been given by Stonely and are available in a standard reference (p.213, Ewing, Jardetsky, and Press). Figures 4 through [illegible] illustrate the variation of the magnitude of the diffraction coefficients which are <u>even</u> functions of the discontinuity angle $\Theta - \pi$. The curves are indexed by four values of the dimensionless parameter $k_1 H$, namely 5,6,7,8; if $k_1 H = 5$, then the second mode is just above cut-off, and if $k_1 H = 8$, the third mode is just below cut-off. It is very interesting to note that the conversion coefficient T_{12} exceeds the reflection coefficient. That is, there is a tendency for the energy to continue to propagate in the same direction even if it necessitates a transfer of modal characteristics.

2. <u>Interpretation</u>

If we compare any two waves of identical characteristics, then their relative energy is proportional to the absolute square of any corresponding amplitude. On the other hand, before we can compare the energy in the fundamental $\mathcal{J}_1$-wave to that of an $\mathcal{J}_2$-wave, its first harmonic, we need make some further calculations. Without loss in generality, let us specialize our discussion to the horizontal wedge face for which the $\mathcal{J}_i$-waves are given explicitly by (2.5) and (2.6), and evaluate the scalar product along the wavefront $y = 0$. Thus $|A_i|^2 \| \mathcal{J}_i \|^2$ represents

Figure 4: The magnitude of the Love wave diffraction coefficients for $\lambda_1 H = 5.$; they are even functions of the discontinuity angle $\Theta - \pi$. In this case $k_2/k_1 = 1.297$, $\mu_2/\mu_1 = 2.159$; the normalization value $N = 2.0$.

Figure 5: The magnitude of the Love wave diffraction coefficients for $\lambda_1 H = 6.$; they are even functions of the discontinuity angle $\Theta - \pi$. In this case $k_2/k_1 = 1.297$, $\mu_2/\mu_1 = 2.159$; the normalization value $N = 1.2$.

Figure 6: The magnitude of the Love wave diffraction coefficients for $\lambda_1 H = 7.$; they are even functions of the discontinuity angle $\Theta - \pi$. In this case $k_2/k_1 = 1.297$, $\mu_2/\mu_1 = 2.159$; the normalization value $N = 1.1$.

Figure 7: The magnitude of the Love wave diffraction coefficients for $\lambda_1 H = 8.$; they are even functions of the discontinuity angle $\Theta - \pi$. In this case $k_2/k_1 = 1.297$, $\mu_2/\mu_1 = 2.159$; the normalization value $N = 1.0$.

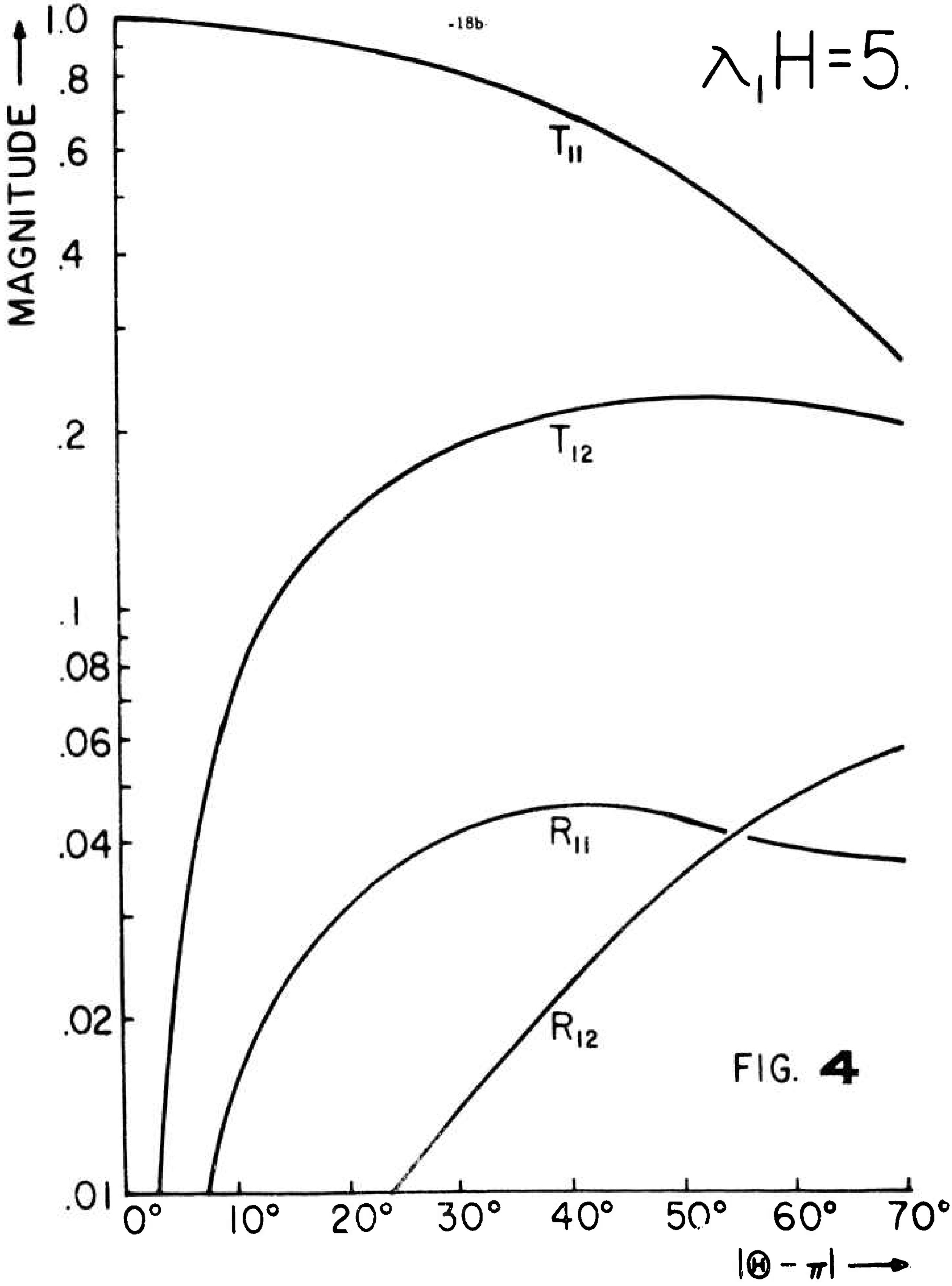

$\lambda_1 H = 5.$
MAGNITUDE
1.0
.8
.6
.4
.2
.1
.08
.06
.04
.02
.01
T_{11}
T_{12}
R_{11}
R_{12}
0°
10°
20°
30°
40°
50°
60°
70°
$|\Theta - \pi|$

FIG. 4

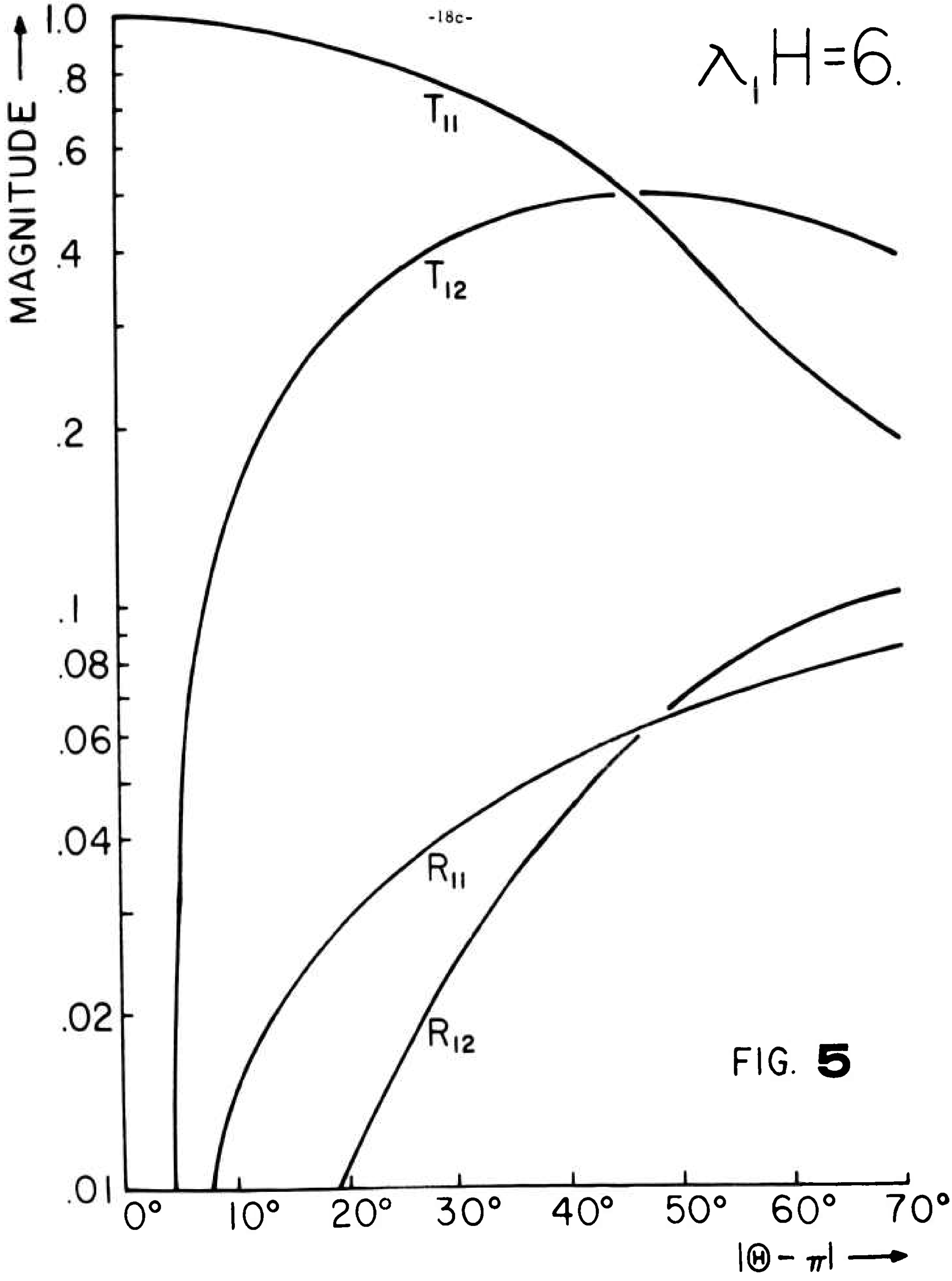

$\lambda_1 H = 6.$
MAGNITUDE
1.0
.8
.6
.4
.2
.1
.08
.06
.04
.02
.01
T_{11}
T_{12}
R_{11}
R_{12}
0°
10°
20°
30°
40°
50°
60°
70°
$|\Theta - \pi|$

FIG. **5**

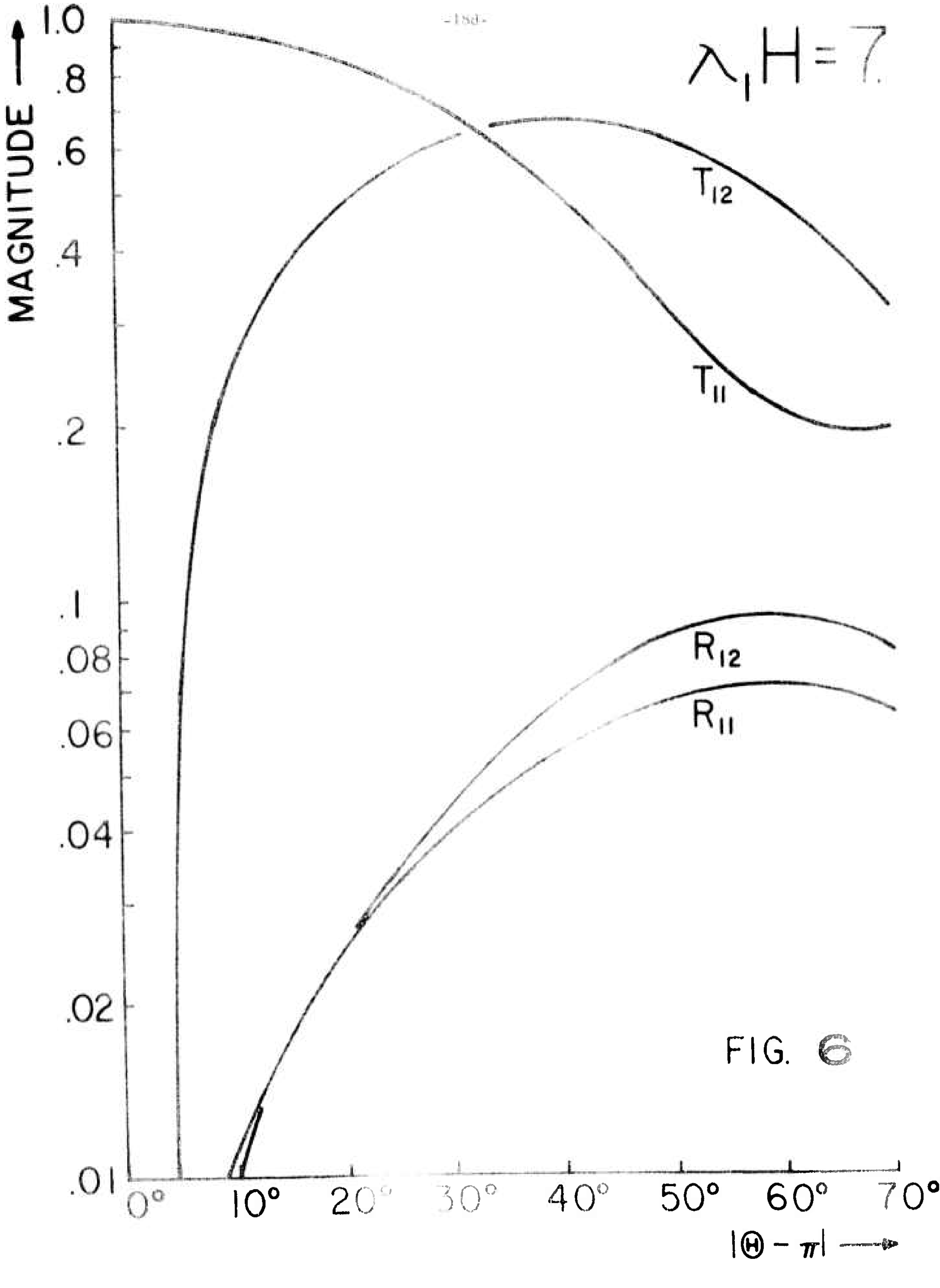
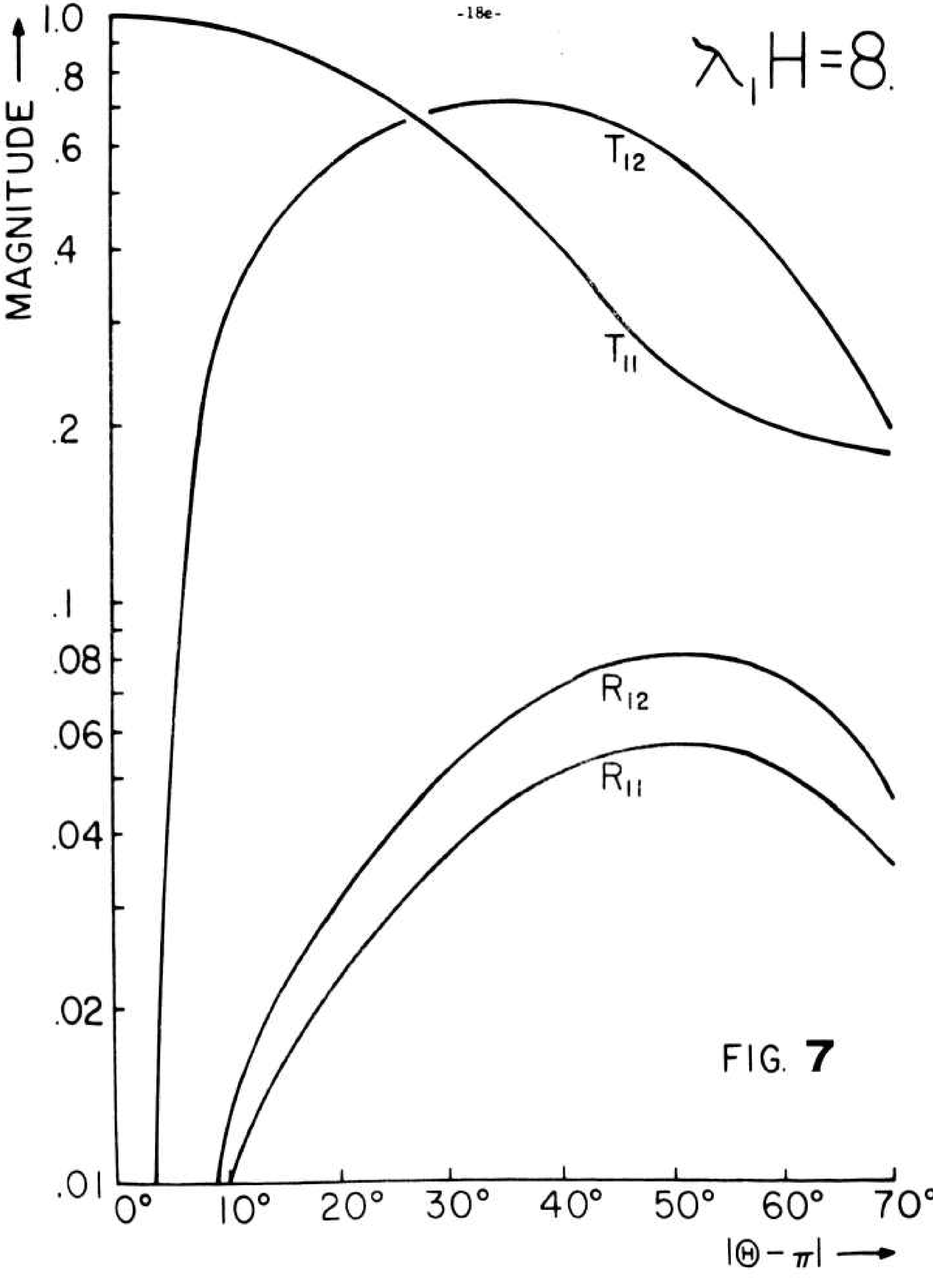

$\lambda_1 H = 7$
MAGNITUDE
1.0
.8
.6
.4
.2
.1
.08
.06
.04
.02
.01
T_{12}
T_{11}
R_{12}
R_{11}
0°
10°
20°
30°
40°
50°
60°
70°
$|\Theta - \pi|$

FIG. 6



$\lambda_1 H = 8.$
MAGNITUDE
1.0
.8
.6
.4
.2
.1
.08
.06
.04
.02
.01
T_{12}
T_{11}
R_{12}
R_{11}
0°
10°
20°
30°
40°
50°
60°
70°
$|\Theta - \pi|$

FIG. **7**

the mean square energy flux transported by an $\mathcal{E}_i$-wave of amplitude A_i. If we denote the group velocity of an $\mathcal{E}_i$-wave as v_i it follows that the ratio

$$\frac{|A_1|^2 \, v_1 \, \| \mathcal{E}_1 \|^2}{|A_2|^2 \, v_2 \, \| \mathcal{E}_2 \|^2} \tag{2.24}$$

compares the power flow of an $A_1\mathcal{E}_1$-wave to an $A_2\mathcal{E}_2$-wave. In particular, a mode near cut-off behaves like an unbounded plane wave in the E_2-medium; hence such a wave can carry large amounts of power even if its amplitude is deceptively small. As a result, if we are to discuss power transfer, we should renormalize the amplitudes of the conversion coefficients

$$R^N_{12} = NR_{12}, \qquad T^N_{12} = NT_{12} \tag{2.25}$$

where

$$N^2 = \frac{v_2}{v_1} \frac{\| \mathcal{E}_2 \|}{\| \mathcal{E}_1 \|} \tag{2.26}$$

so that $|R^N_{12}|^2$ and $|T^N_{12}|^2$ are proportional to the power transferred by the diffraction of a $\mathcal{E}_1$-wave of unit amplitude at a wedge discontinuity. The appropriate values of N for the previous numerical example are cited in the captions of Figures 4 - 7.

In a fashion similar to the error analysis of Part I, the function $[1-|R_{11}|^2-|R^N_{12}|^2-|T_{11}|^2-|T^N_{12}|^2]$ represents the amount of ambiguous energy. These values are more satisfactory in the present analysis than in Part I; this can be explained by the fact that we have a more flexible trial function since we can vary the coefficients of <u>two</u> reflected modes.

APPENDIX

Love wave diffraction coefficients would be very difficult to measure in the laboratory, but the techniques of two-dimensional model seismology offer a means of determining Rayleigh wave reflection and transmission coefficients with an accuracy of about 10-20 per cent. The present theory and experiment agree if $\Theta \sim \pi$, but outside this range, there are experimental features which are not duplicated by the results of the present elementary variational procedure. The analysis in the harmonic domain could be refined by employing various devices to reduce the amount of unexplained energy. Such calculations would probably require substantial effort, and the idealized formulation of the present problem should be reviewed if the labor is to have relevance to pulse measurements.

There are major distinctions between analysis in the harmonic and time domain. For example, whereas a harmonic Rayleigh wave is a uniquely defined entity, Friedlander (1948) has pointed out that a Rayleigh pulse can assume a variety of waveforms. Furthermore, any Rayleigh pulse can not have a sharply defined wavefront, and theoretically must give infinite advance notice of its arrival, unless it merges continuously with a precursor, typically the shear psuedo-surface wave (Cagniard 1939). Although the amplitude of this shear wave decays with distance, <u>its integrated flux remains constant</u>. It is difficult

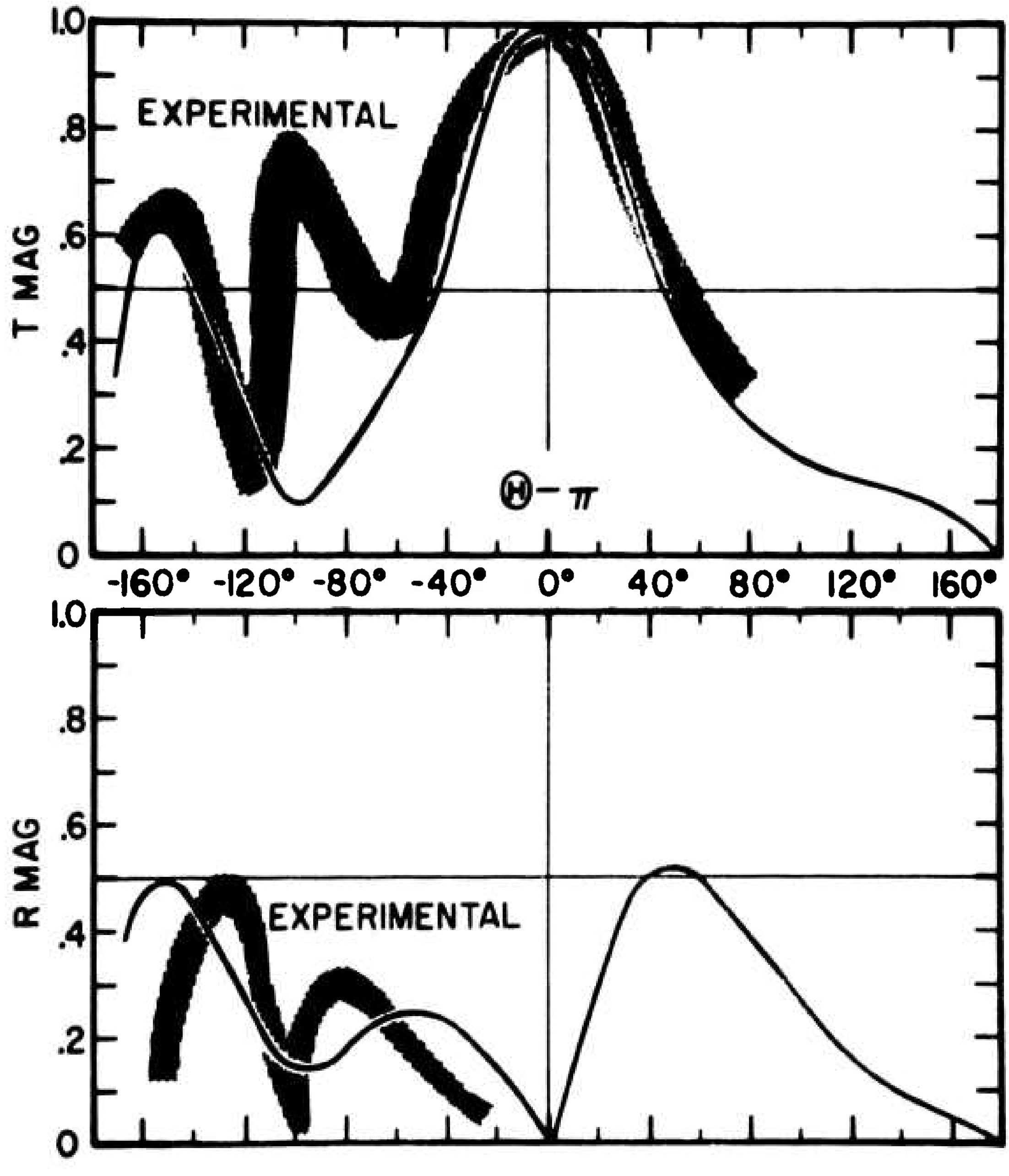


Figure 8: The diffraction coefficients R and T for a trial function consisting of an incident and reflected Rayleigh wave. However, the shear coefficient Γ of the incident Rayleigh wave has been incremented by a factor $(1 + \cos \Theta + \cos^2 \Theta)$. In this case, Poisson's ratio $\sigma = 1/4$.

to separate the far-field effects due to the arrival of a Rayleigh pulse and its shear companion at the second wedge face. In other words, in addition to Rayleigh/Rayleigh interactions, there will be some shear/Rayleigh conversions. What contributions might this shear wave introduce? Whereas we can not give a rigorous answer to this question, we can however make a rough, but simple, estimate.

We first note that if the wedge angle Θ is π or $\pi/2$, then we would expect little or no shear/Rayleigh conversion. In the first case, there is no discontinuity, and the second case corresponds to a geometry for which the shear wave is essentially normal to the second wedge face, and we know that for normal incidence, a shear wave is reflected as a shear wave. Then, from Equation (1.6), we note that we can easily add some additional shear potential to the original excitation by incrementing the Rayleigh wave's shear coefficient Γ by an additional contribution $f(\Theta)$ depending on the wedge angle

$$\Gamma' = \Gamma \left[1 + f(\Theta) \right]$$

Quite arbitrarily, we have chosen

$$f(\Theta) = \cos\Theta + \cos^2\Theta$$

whose sole merit is that it is the simplest function we could think of that vanishes for $\Theta = \pi, \pi/2$. It is then a trivial matter to repeat the calculations appropriate for Figure 2, and the results are plotted in Figure 8. The shaded area indicates the range of experimental points as measured by Knopoff and Gangi(1960), deBremaecker(1958). and Viktarov, (1958). Of course there is limited justification for this heuristic procedure, but it is remarkable that with this naive device the coefficients R and T adopt many of the characteristics of the experimental data. In any event, we conclude that more refined calculations should use a more realistic excitation.

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